

ASYMPTOTIC ARBITRAGE IN THE HESTON MODEL

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ABSTRACT. In the context of the Heston model, we establish a precise link between the set of equivalent martingale measures, the ergodicity of the underlying variance process and the concept of asymptotic arbitrage proposed in Kabanov-Kramkov [13] and in Föllmer-Schachermayer [8].

1. INTRODUCTION

The concept of arbitrage is the cornerstone of modern mathematical finance, and several versions of the so-called fundamental theorem of asset pricing have been proved over the past two decades, see for instance [5] for an overview. A version of it essentially states that absence of arbitrage is equivalent to the existence of an equivalent martingale measure under which discounted asset prices are true martingales. This then allows the use of ‘martingale models’ (either continuous or with jumps) as underlying dynamics for option pricing. In practice, should short-term arbitrages arise—due to some market discrepancies—they are immediately exploited by traders, and market liquidity therefore acts as an equilibrium agent, to prevent them occurring significantly. It can be argued, however, that one may generate long-term riskless profit, when the time horizon tends to infinity. This turns out to hold in most models used in practice. The existence and nature of such infinite horizon asymptotic arbitrage opportunities have been studied in a handful of papers, for example [7, 14, 19].

Among the plethora of models used and analysed both in practice and in theory, stochastic volatility models have proved to be very flexible and suitable for pricing and hedging. Due to its affine structure, the Heston model [11] has gained great popularity among practitioners for equity and FX derivatives modelling, see in particular [10, 9] for a detailed account of this fame. Because of the correlation between the asset price and the underlying volatility, the market is incomplete, and the Heston model admits an infinity of equivalent martingale measures. Its affine structure allows us to study precisely the existence (or absence) of asymptotic arbitrage. Specifically, we shall endeavour to understand how the parameters of the model influence the nature—such as its speed and existence—of the asymptotic arbitrage. Of particular interest will be the link between asymptotic arbitrage and the ergodicity of the underlying variance process. In [8] the authors proved under suitable regularity conditions that price processes with a non-trivial market price of risk (see Definition 2.3) allow for asymptotic arbitrage (with linear speed). Using the theory of large deviations, we shall show that S may allow for such arbitrage even if it does not admit an average squared market price of risk.

The organisation of this paper is as follows: all the notations and definitions are given in Section 2. Asymptotic arbitrage in the Heston model is studied in Section 3; the main contribution of this paper is Theorem 3.7, which identifies sufficient (and sometimes necessary) conditions on the set of equivalent

Date: February 27, 2013.

Key words and phrases. Stochastic volatility model, Heston model, asymptotic arbitrage, large deviations.

martingale measures under which asymptotic arbitrage occur with linear speed. These conditions are different from those in Proposition 3.11 in which we study how the ergodicity of the variance process plays an important role to prove the existence of asymptotic arbitrage with slower speed.

2. NOTATIONS AND DEFINITIONS

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions and let $S = e^X$ model a risky security under an equivalent martingale measure. Let \mathcal{H} denote the class of predictable, S -integrable admissible processes. We define for each $t > 0$ the sets $K_t := \left\{ \int_0^t H_s dS_s : H \in \mathcal{H} \right\}$ and $\mathcal{M}_t^e(S) := \{ \mathbb{Q} \sim \mathbb{P} \text{ such that } (S_u)_{0 \leq u \leq t} \text{ is a local } \mathbb{Q}\text{-martingale} \}$. We shall always assume that $\mathcal{M}_t^e(S)$ is not empty for all $t \geq 0$. Furthermore, for any set A in Ω , we shall denote by $A^c := \Omega \setminus A$ its complement.

2.1. Asymptotic arbitrage. The following definition of a long-term arbitrage is taken from [8]:

Definition 2.1. The process S admits an $(\varepsilon_1, \varepsilon_2)$ -arbitrage up to time t if for $(\varepsilon_1, \varepsilon_2) \in (0, 1)^2$, there exists $X_t \in K_t$ such that

- (i) $X_t \geq -\varepsilon_2$ \mathbb{P} -almost surely;
- (ii) $\mathbb{P}(X_t \geq 1 - \varepsilon_2) \geq 1 - \varepsilon_1$.

This means that the maximal loss of the trading strategy, yielding the wealth X_t at time t , is bounded by ε_2 and with probability $1 - \varepsilon_1$ the terminal wealth X_t equals at least $1 - \varepsilon_2$. We shall be interested here in the following characterisation of long-term arbitrage, namely the notion of asymptotic exponential arbitrage with exponentially decaying failure probability, first proposed in [8] and later in [3] and [7].

Definition 2.2. The process S allows for asymptotic exponential arbitrage with exponentially decaying failure probability if there exist $t_0 \in (0, \infty)$ and constants $C, \lambda_1, \lambda_2 > 0$ such that for all $t \geq t_0$, there is $X_t \in K_t$ satisfying

- (i) $X_t \geq -e^{-\lambda_2 t}$ \mathbb{P} -almost surely;
- (ii) $\mathbb{P}(X_t \leq e^{\lambda_2 t}) \leq Ce^{-\lambda_1 t}$.

Asymptotic exponential arbitrage with exponentially decaying failure probability can be interpreted as a strong and quantitative form of long-term arbitrage. In particular, let $\lambda_1, \lambda_2 > 0$, $\varepsilon_2 := e^{-\lambda_2 t}$ and $\varepsilon_1 := Ce^{-\lambda_1 t}$, then Definitions 2.1 and 2.2 are equivalent. In [8], Föllmer and Schachermayer showed that this strong form of asymptotic arbitrage was actually a consequence, under some assumptions (see Theorem 1.4 therein), of the following concept:

Definition 2.3. Let $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a smooth function such that $\lim_{t \nearrow +\infty} f(t) = +\infty$. The process S is said to have an average squared market price of risk γ_i ($i = 1, 2$) above the threshold $c_i > 0$ with speed $f(t)$ if $\mathbb{P} \left(f(t)^{-1} \int_0^t \gamma_i^2(s) ds < c_i \right)$ tends to zero as t tends to infinity.

2.2. Stochastic volatility models. We consider here the Heston stochastic volatility model, namely the unique strong solution to the stochastic differential equations (2.1) below. As is well-known (see [12] for example), there may not be a unique risk-neutral martingale measure for such models. The following

SDEs are therefore understood under one such risk-neutral measure \mathbb{Q} .

$$(2.1) \quad \begin{aligned} dS_t/S_t &= \mu dt + \sqrt{V_t} \left(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \right), & S_0 &= 1 \\ dV_t &= (a - bV_t)dt + \sqrt{2\sigma V_t} dW_1(t), & V_0 &> 0, \end{aligned}$$

where W_1 and W_2 are independent \mathbb{Q} -Brownian motions, $\mu, a, \sigma > 0$, $b \in \mathbb{R}$ and $|\rho| < 1$. The class of equivalent martingale measures \mathbb{Q} can be considered in terms of the Radon-Nikodym derivatives

$$(2.2) \quad Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left\{ - \left(\int_0^t \gamma_1(s) dW_1(s) + \int_0^t \gamma_2(s) dW_2(s) \right) - \frac{1}{2} \left(\int_0^t \gamma_1^2(s) ds + \int_0^t \gamma_2^2(s) ds \right) \right\}.$$

The condition $\mu - r = \sqrt{V_t} \left(\rho \gamma_1(t) + \sqrt{1-\rho^2} \gamma_2(t) \right)$ is necessary for an equivalent local martingale measure to exist, and ensures that the discounted stock price is a local martingale. Since Z is a positive local martingale with $Z_0 = 1$, it is a supermartingale, and a true martingale if and only if $\mathbb{E}(Z_t) = 1$. For the Heston stochastic volatility model we obtain, for any real constant λ ,

$$(2.3) \quad \gamma_1(t) = \lambda \sqrt{V_t} \quad \text{and} \quad \gamma_2(t) = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{\mu - r}{\sqrt{V_t}} - \lambda \rho \sqrt{V_t} \right).$$

3. MAIN RESULTS

For any $(\alpha, \beta, \delta) \in \mathbb{R}^3$, we introduce the process $(X_t^{\alpha, \beta, \delta})_{t \geq 0}$ defined (pathwise) by

$$(3.1) \quad X_t^{\alpha, \beta, \delta} := \alpha V_t + \beta \int_0^t V_s ds + \delta \int_0^t V_s^{-1} ds, \quad \text{for any } t \geq 0,$$

where V is the Feller diffusion for the variance in (2.1). Define the real interval $\mathcal{D}_{\beta, \delta}$ by

$$(3.2) \quad \mathcal{D}_{\beta, \delta} = \begin{cases} \left[\frac{(a-\sigma)^2}{4\sigma\delta}, \frac{b^2}{4\sigma\beta} \right], & \text{if } \beta > 0, \delta < 0, \\ \left(-\infty, \frac{(a-\sigma)^2}{4\sigma\delta} \wedge \frac{b^2}{4\sigma\beta} \right], & \text{if } \beta > 0, \delta > 0, \\ \left[\frac{b^2}{4\sigma\beta}, \frac{(a-\sigma)^2}{4\sigma\delta} \right], & \text{if } \beta < 0, \delta > 0, \\ \left(\frac{(a-\sigma)^2}{4\sigma\delta} \vee \frac{b^2}{4\sigma\beta}, +\infty \right), & \text{if } \beta < 0, \delta < 0. \end{cases}$$

Whenever $\beta\delta = 0$, we define $\mathcal{D}_{\beta, \delta}$ by taking the limits of the interval (a closed bound becoming open if it becomes infinite), where we use the slight abuse of notation " $1/0 = \infty$ ", i.e. $\mathcal{D}_{\beta, \delta} = \left(-\infty, \frac{b^2}{4\sigma\beta} \right]$ if $\beta > 0$ and $\delta = 0$, $\mathcal{D}_{\beta, \delta} = \left[\frac{b^2}{4\sigma\beta}, +\infty \right)$ if $\beta < 0$ and $\delta = 0$, and $\mathcal{D}_{\beta, \delta} = \mathbb{R}$ if $\beta = \delta = 0$. Let us further define the function $\Lambda^{\beta, \delta} : \mathcal{D}_{\beta, \delta} \rightarrow \mathbb{R}$ by

$$(3.3) \quad \Lambda^{\beta, \delta}(u) = \begin{cases} \frac{ba}{2\sigma} - \frac{1}{2\sigma} \sqrt{((a-\sigma)^2 - 4\sigma\delta u)(b^2 - 4\sigma\beta u)} - \frac{1}{2} \sqrt{b^2 - 4\sigma\beta u}, & \text{if } \delta \neq 0, \\ \frac{a}{2\sigma} \left(b - \sqrt{b^2 - 4\sigma\beta u} \right), & \text{if } \delta = 0. \end{cases}$$

In the case $\delta \neq 0$ above, we further impose the condition $a > \sigma$ for the definition of the function $\Lambda^{\beta, \delta}$.

Remark 3.1. It may be surprising at first that the function $\Lambda^{\beta, \delta}$ related—in some sense defined precisely below—does not depend on α . This function actually describes the large-time behaviour of the process $X^{\alpha, \beta, \delta}$. Since the variance process V is strictly positive almost surely (by the Feller condition imposed above), the term $\int_0^t V_s ds$ clearly dominates V_t for any t , which explains why α bears no influence on $\Lambda^{\beta, \delta}$.

The condition $a > \sigma$ imposed above in the case $\delta \neq 0$ should not surprise the reader since this is nothing else than the Feller condition, ensuring that the variance process never touches the origin almost surely.

We further define the Fenchel-Legendre transform $\Lambda_{\beta,\delta}^* : \mathbb{R} \rightarrow \mathbb{R}_+$ of $\Lambda^{\beta,\delta}$ by

$$(3.4) \quad \Lambda_{\beta,\delta}^*(x) := \sup_{u \in \mathcal{D}_{\beta,\delta}} \{ux - \Lambda^{\beta,\delta}(u)\}.$$

Notation. Whenever $\beta = 0$ or $\delta = 0$, we shall drop the subscript and write respectively Λ^δ or Λ^β . The same rule will be followed for the domains and the Fenchel-Legendre transforms.

In the general case, $\Lambda_{\beta,\delta}^*$ does not have a closed-form representation. In the particular case where δ is null—which shall be of interest for us—it actually does, and a straightforward computation shows that

$$(3.5) \quad \Lambda_\beta^*(x) = \frac{(bx - a\beta)^2}{4\sigma|\beta x|}, \quad \text{for all } x \in \mathbb{R}^*.$$

In that case, the function Λ_β^* is strictly convex on \mathbb{R}_+^* (respectively on \mathbb{R}_-^*) with a unique minimum attained at $|a\beta/b|$ (resp. at $-|a\beta/b|$). In particular on \mathbb{R}_+^* , if $b\beta \leq 0$ then Λ_β^* is strictly decreasing and strictly positive on \mathbb{R}_+^* . Otherwise, if $b\beta > 0$, then $\Lambda_\beta^*(|a\beta/b|) = 0$ and $\Lambda_\beta^*(x) > 0$ for all $x \in \mathbb{R}_+^* \setminus \{|a\beta/b|\}$. Symmetric statements hold on \mathbb{R}_-^* .

3.1. The large deviations case. In this section, we shall be interested in proving asymptotic arbitrage results for the stock price process when the speed is linear. We shall in particular observe that the ergodicity of the variance process plays a key role. We first start though with the following two technical lemmas, which will be used heavily in the remaining of the paper, and the proofs of which can be found in Appendix A.

Lemma 3.2. *For any $(\alpha, \beta) \in \mathbb{R}^2$, the family $(t^{-1}X_t^{\alpha,\beta,0})_{t>0}$ satisfies a large deviations principle on \mathbb{R}_+^* if $\beta > 0$ and on \mathbb{R}_-^* if $\beta < 0$ with speed t^{-1} and rate function Λ_β^* characterised in (3.5).*

Lemma 3.3. *The family $(t^{-1}X_t^{\alpha,\beta,\delta})_{t \geq 0}$ satisfies*

- (i) *a full LDP (on \mathbb{R}) if $\beta\delta < 0$;*
- (ii) *a partial LDP on $(2\sqrt{\delta\beta}, +\infty)$ if $\beta > 0$ and $\delta > 0$;*
- (iii) *a partial LDP on $(-\infty, -2\sqrt{\delta\beta})$ if $\beta < 0$ and $\delta < 0$;*
- (iv) *a partial LDP if $\beta = 0$ or $\delta = 0$ on the domain given by taking the limit in (ii) or (iii).*

In each case, the rate function is $\Lambda_{\beta,\delta}^$ and the (partial) LDP holds with speed t^{-1} .*

In [8, Theorem 1.4], Föllmer and Schachermayer proved that if the stock price process has an average market price of risk above a threshold then asymptotic arbitrage holds. Using the large deviations principle proved above, we first show that S does not always admit an average market price of risk for γ_1 (Proposition 3.4) or γ_2 (Proposition 3.5) above any threshold. This is in particular so when the variance process is not ergodic ($b \leq 0$). This however—as proved in Theorem 3.7 below—does not preclude absence of asymptotic arbitrage.

Proposition 3.4. *Fix $\lambda \geq 0$ and $c > 0$. The stock price process does not satisfy an average squared market price of risk γ_1 above the threshold c if either (i) $b \leq 0$ or (ii) $b > 0$ and $c > a\lambda^2/b$.*

Proof. Note first that $\lambda = 0$ implies $\gamma_1 \equiv 0$ and hence $\mathbb{P}\left(t^{-1} \int_0^t \gamma_1^2(s) ds < c\right) = 1$ for all $t > 0$, so that the proposition is trivial. Assume from now on that $\lambda \neq 0$ and let c be an arbitrary strictly positive real number. The definition of γ_1 in (2.3) implies $\mathbb{P}(t^{-1} \int_0^t \gamma_1^2(s) ds \geq c) = \mathbb{P}(t^{-1} \int_0^t V_s ds \geq c/\lambda^2) = \mathbb{P}(t^{-1} X_t^{0,1} \geq c/\lambda^2)$. From Lemma 3.2, the family $(t^{-1} X_t^{0,1})_{t \geq 0}$ satisfies a LDP on \mathbb{R}_+^* with rate function $\Lambda_{1,0}^*$. Hence

$$\limsup_{t \nearrow +\infty} \frac{1}{t} \log \mathbb{P}\left(X_t^{0,\beta} \geq \frac{c}{\lambda^2}\right) \leq - \inf_{\{x \geq c/\lambda^2\}} \Lambda_{1,0}^*(x) = \begin{cases} -\Lambda_{1,0}^*(c/\lambda^2) < 0, & \text{if } c > a\lambda^2/|b|, \\ 0, & \text{if } c \leq a\lambda^2/|b|, \end{cases}$$

When $b \leq 0$, $\Lambda_{1,0}^*(c/\lambda^2)$ is strictly positive for all $c > 0$. Thus $\mathbb{P}(t^{-1} X_t^{0,1} \geq c/\lambda^2)$ converges to zero as t tends to infinity, which in turn implies that $\mathbb{P}(t^{-1} \int_0^t \gamma_1^2(s) ds < c)$ converges to 1 as t tends to infinity, and statement (i) in the proposition follows. When $b > 0$, consider the case $c > a\lambda^2/b$. There exists $\bar{t} > 0$ such that for all $t \geq \bar{t}$, $\mathbb{P}(t^{-1} X_t^{0,1} \geq c/\lambda^2) \leq \exp(-\Lambda_{1,0}^*(c/\lambda^2)t)$, and hence $\mathbb{P}(t^{-1} X_t^{0,1} \geq c/\lambda^2)$ converges to zero as t tends to infinity, which again proves statement (ii) in the proposition. \square

Proposition 3.5. *Fix $\lambda \geq 0$ and let $c > 0$. The stock price process does not satisfy an average squared market price of risk γ_2 above the threshold c if any of the following conditions hold:*

- (i) $\lambda\rho(\mu - r) > 0$;
- (ii) $\lambda\rho(\mu - r) < 0$ and $c > -4\lambda\rho(\mu - r)/(1 - \rho^2)$;
- (iii) $\lambda\rho \neq 0$, $\mu = r$ and $b \leq 0$;
- (iv) $\lambda\rho \neq 0$, $\mu = r$, $b > 0$ and $c > a\lambda^4\rho^2/(b(1 - \rho^2))$;
- (v) $\lambda\rho = 0$;

Remark 3.6. Note that the case of a complete market ($\rho = 0$) is included in case (v) of the proposition.

Proof. Let c be an arbitrary strictly positive real number. Note first that if $\lambda\rho = 0$, and $\mu = r$, then $\gamma_2 \equiv 0$ and hence $\mathbb{P}\left(t^{-1} \int_0^t \gamma_2^2(s) ds < c\right) = 1$ for all $t > 0$. If $\mu \neq r$, then

$$\mathbb{P}\left(\frac{1}{t} \int_0^t \gamma_2^2(s) ds \geq c\right) = \mathbb{P}\left(\frac{1}{t} \int_0^t \frac{ds}{V_s} \geq \frac{1 - \rho^2}{(\mu - r)^2} c\right) = \mathbb{P}\left(\frac{X_t^{0,0,1}}{t} \geq \frac{1 - \rho^2}{(\mu - r)^2} c\right),$$

and Lemma A.1 implies that $\Lambda_{0,1}^*$ is strictly positive, so that (v) follows. Assume now that $\lambda\rho \neq 0$ and $\mu \neq r$. The definition of γ_2 in (2.3) implies that

$$\begin{aligned} \mathbb{P}\left(\frac{1}{t} \int_0^t \gamma_2^2(s) ds \geq c\right) &= \mathbb{P}\left(\frac{(\mu - r)^2}{1 - \rho^2} \frac{1}{t} \int_0^t \frac{ds}{V_s} + \frac{\lambda^2 \rho^2}{1 - \rho^2} \frac{1}{t} \int_0^t V_s ds \geq c + \frac{2\rho\lambda(\mu - r)}{1 - \rho^2}\right) \\ &= \mathbb{P}\left(\frac{X_t^{0,\beta,\delta}}{t} \geq c + \frac{2\rho\lambda(\mu - r)}{1 - \rho^2}\right), \end{aligned}$$

where $\beta = \frac{(\mu - r)^2}{1 - \rho^2} > 0$, $\delta = \frac{\lambda^2 \rho^2}{1 - \rho^2} > 0$, and where $X^{0,\beta,\delta}$ is defined in (3.1). By Lemma 3.3, the family $(X_t^{0,\beta,\delta}/t)_{t > 0}$ satisfies a large deviations principle on $(2\sqrt{\delta\beta}, +\infty)$ with rate function $\Lambda_{\beta,\delta}^*$, i.e.

$$\limsup_{t \nearrow +\infty} t^{-1} \log \mathbb{P}\left(\frac{1}{t} \int_0^t \gamma_2^2(s) ds \geq c\right) \leq - \inf \left\{ \Lambda_{\beta,\delta}^*(x) : x \geq c + \frac{2\rho\lambda(\mu - r)}{1 - \rho^2} \right\}.$$

When $\lambda\rho(\mu - r) > 0$, $\left[c + \frac{2\rho\lambda(\mu - r)}{1 - \rho^2}, +\infty\right)$ is a subset of $(2\sqrt{\delta\beta}, +\infty)$, and (i) follows immediately from Lemma A.1. When $\lambda\rho(\mu - r) < 0$, the interval $\left[c + \frac{2\rho\lambda(\mu - r)}{1 - \rho^2}, +\infty\right)$ is a subset of $(2\sqrt{\delta\beta}, +\infty)$ if and only if $c > -\frac{4\rho\lambda(\mu - r)}{1 - \rho^2} > 0$. Since $\beta\delta = \frac{\lambda^2 \rho^2 (\mu - r)^2}{(1 - \rho^2)^2} > 0$, Lemma A.1 implies that $\Lambda_{\beta,\delta}^*(x) > 0$ for any

$x > 2\sqrt{\beta\delta} = \frac{2|\lambda\rho(\mu-r)|}{1-\rho^2}$. Therefore, $\mathbb{P}\left(X_t^{0,\beta,\delta}/t \geq c + \frac{2\rho\lambda(\mu-r)}{1-\rho^2}\right)$ converges to zero as t tends to infinity. Then $\mathbb{P}(t^{-1} \int_0^t \gamma_2^2(s)ds < c)$ converges to one as t tends to infinity. Assume that $\lambda\rho \neq 0$ and $\mu = r$. The definition of γ_2 in (2.3) implies that

$$\mathbb{P}\left(\frac{1}{t} \int_0^t \gamma_2^2(s)ds \geq c\right) = \mathbb{P}\left(\frac{1}{t} \int_0^t V_s ds \geq \frac{1-\rho^2}{\lambda^2\rho^2}c\right) = \mathbb{P}\left(\frac{X_t^{0,1,0}}{t} \geq \frac{1-\rho^2}{\lambda^2\rho^2}c\right),$$

and (iii) and (iv) then from Proposition 3.4. \square

We can now move on to our main theorem.

Theorem 3.7. *Let $\varepsilon \in (0,1)$, $\gamma > 0$ and define the set $A_{\lambda,t} := \{Z_t \geq e^{-\gamma t}\} \in \mathcal{F}_t$. Then S allows for strong asymptotic arbitrage (with speed t) with exponentially decaying probability (in the sense of Definition 2.2) with $C = \exp(\lambda V_0/\sqrt{2\sigma})$, $\lambda_1 = -\left(\frac{a\lambda}{\sqrt{2\sigma}} + \gamma + \Lambda^{\alpha,\beta}(1)\right)$ and $\lambda_2 = \gamma$ if*

- (i) $\lambda \in \mathbb{R} \setminus \left(-\frac{b}{\sqrt{2\sigma}} - \frac{\gamma}{a\zeta_+}, -\frac{b}{\sqrt{2\sigma}} + \frac{\gamma}{a\zeta_-}\right)$, when $\sqrt{2\sigma} > 1$;
- (ii) $\lambda < -\frac{b}{\sqrt{2\sigma}} - \frac{\gamma}{a\zeta_+}$, when $\sqrt{2\sigma} \leq 1$,

where we define $\zeta_{\pm} := \sqrt{2\sigma} \pm 1/\sqrt{2\sigma}$.

Remark 3.8.

- Note that the sufficient condition is not necessary. Consider for instance $\lambda = 0$ and $\mu = r$. Then clearly $Z_t = 1$ almost surely for all $t \geq 0$, and $\mathbb{P}(Z_t \geq e^{-\gamma t}) = 1$ for any $\gamma > 0$.
- Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function such that $t/f(t)$ tends to infinity as t tends to infinity, then for any $\gamma > 0$ and t large enough, $e^{-\gamma f(t)} \geq e^{-\gamma t}$. Therefore $\mathbb{P}(Z_t \geq e^{-\gamma f(t)})$ tends to zero as well as t tends to infinity; We cannot however conclude that Theorem 3.7 holds, i.e. that S allows asymptotic arbitrage with speed $f(t)$, since this does not give us any information about the behaviour of $\mathbb{Q}(Z_t \geq e^{-\gamma f(t)})$.

Proof. Let $\gamma > 0$ and define the set $A_{\lambda,t} := \{Z_t \geq e^{-\gamma t}\} \in \mathcal{F}_t$. Since the processes W_2 and V are independent, the tower property for conditional expectation implies $\mathbb{E}(Z_t) = \mathbb{E}\left(e^{-\int_0^t \gamma_1(s)dW_1(s) - \frac{1}{2}\int_0^t \gamma_1^2(s)ds}\right)$. Markov's inequality therefore yields

$$\begin{aligned} \mathbb{P}(A_{\lambda,t}) &\leq \frac{\mathbb{E}(Z_t)}{\exp(-\gamma t)} = \frac{\mathbb{E}\left[\exp\left(-\int_0^t \gamma_1(s)dW_1(s) - \frac{1}{2}\int_0^t \gamma_1^2(s)ds\right)\right]}{e^{-\gamma t}} \\ &= \exp\left(\frac{\lambda V_0}{\sqrt{2\sigma}} + \frac{a\lambda t}{\sqrt{2\sigma}} + \gamma t\right) \mathbb{E}\left[\exp\left(-\frac{\lambda V_T}{\sqrt{2\sigma}} - \left(\frac{b\lambda}{\sqrt{2\sigma}} + \frac{\lambda^2}{2}\right)\int_0^t V_s ds\right)\right] \\ &= \exp\left[\frac{\lambda V_0}{\sqrt{2\sigma}} + \left(\frac{a\lambda}{\sqrt{2\sigma}} + \gamma\right)t\right] \Lambda_t^{\alpha,\beta}(t), \end{aligned}$$

where $\alpha = -\frac{\lambda}{\sqrt{2\sigma}}$ and $\beta = -\frac{b\lambda}{\sqrt{2\sigma}} - \frac{\lambda^2}{2}$. From the proof of Lemma 3.2, we know that $t^{-1} \log \Lambda_t^{\alpha,\beta}(t)$ converges to $\Lambda^{\alpha,\beta}(1)$. This implies that for any $\delta > 0$ there exists $\tilde{t} > 0$ such that for any $t > \tilde{t}$, we have

$$e^{(\Lambda^{\alpha,\beta}(1)-\delta)t} \leq \Lambda_t^{\alpha,\beta}(t) \leq e^{(\Lambda^{\alpha,\beta}(1)+\delta)t}.$$

We then deduce that for any $t > \tilde{t}$,

$$\exp\left[\frac{\lambda V_0}{\sqrt{2\sigma}} + \left(\frac{a\lambda}{\sqrt{2\sigma}} + \gamma + \Lambda^{\alpha,\beta}(1) - \delta\right)t\right] \leq \mathbb{P}(Z_t \geq e^{-\gamma t}) \leq \exp\left[\frac{\lambda V_0}{\sqrt{2\sigma}} + \left(\frac{a\lambda}{\sqrt{2\sigma}} + \gamma + \Lambda^{\alpha,\beta}(1) + \delta\right)t\right].$$

Since δ can be chosen as small as desired, we simply need to prove that $\frac{a\lambda}{\sqrt{2\sigma}} + \gamma + \Lambda^{\alpha,\beta}(1) < 0$. Now,

$$\Lambda^{\alpha,\beta}(1) = \frac{ab}{2\sigma} - a\sqrt{b^2 - 4\sigma\beta} = \frac{ab}{2\sigma} - a\sqrt{b^2 + 4\sigma\lambda\left(\frac{b}{\sqrt{2\sigma}} + \frac{\lambda}{2}\right)} = \frac{ab}{2\sigma} - a\left|\lambda\sqrt{2\sigma} + b\right|,$$

which is always well defined. Therefore, we are left to prove that $|\lambda\sqrt{2\sigma} + b| > \frac{\gamma}{a} + \frac{b}{2\sigma} + \frac{\lambda}{\sqrt{2\sigma}}$. This is a piecewise linear inequality in λ , which is clearly satisfied if and only if

- (i) $\lambda \in \mathbb{R} \setminus \left(-\frac{b}{\sqrt{2\sigma}} - \frac{\gamma}{a\zeta_+}, -\frac{b}{\sqrt{2\sigma}} + \frac{\gamma}{a\zeta_-}\right)$, when $\sqrt{2\sigma} > 1$;
- (ii) $\lambda < -\frac{b}{\sqrt{2\sigma}} - \frac{\gamma}{a\zeta_+}$, when $\sqrt{2\sigma} \leq 1$.

In the first case, the interval is never empty. Let $(\varepsilon_1, \varepsilon_2) := \left(\exp\left[\frac{\lambda V_0}{\sqrt{2\sigma}} + \left(\frac{a\lambda}{\sqrt{2\sigma}} + \gamma + \Lambda^{\alpha,\beta}(1)\right)t\right], e^{-\gamma t}\right)$. Define now the probability measure \mathbb{Q} via the Radon-Nikodym theorem by $\mathbb{Q}(B) := \mathbb{E}(BZ_t)$, for any $B \in \Omega$. Clearly $\mathbb{Q} \in \mathcal{M}_t^e(S)$ and therefore there exists $t_0 > 0$ such that for any $t \geq t_0$, $A_{t,\lambda}$ satisfies $\mathbb{P}(A_{t,\lambda}) \leq \varepsilon_1$ and $\mathbb{Q}(A_{t,\lambda}) \geq 1 - \varepsilon_2$. Proposition 2.1 in [8] implies that S allows for $(\varepsilon_1, \varepsilon_2)$ -arbitrage in the sense of Definition 2.1. Arbitrage with decaying failure as in the theorem immediately follows. \square

3.2. Case $t/f(t)$ tends to infinity as t tends to infinity. Let $b > 0$, in which case the variance process is ergodic and its stationary distribution π is a Gamma law with shape parameter a/σ and scale parameter σ/b ; namely $t^{-1} \int_0^t h(V_s)ds$ converges to $\int_{\mathbb{R}} h(x)\pi(dx)$ almost surely for any $h \in L^1(\pi)$ (see [4] and [15]). In this section, we consider a continuous function $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ such that $t/f(t)$ tends to infinity as t tends to infinity. We shall prove below that (under some conditions on the risk parameter λ) the ergodicity of the variance ensures that S allows an asymptotic arbitrage with sublinear speed $f(t)$.

Proposition 3.9. *The stock price process S in (2.1) has an average squared market price of risk γ_1 above the threshold $a\lambda^2/b$ with speed $f(t)$. If furthermore $a > \sigma$ and $\lambda\rho(\mu - r) \leq 0$, then there exists $c_2 > 0$ such that S has an average squared market price of risk γ_2 above the threshold c_2 with speed $f(t)$.*

Remark 3.10. As the proof shows, we can actually be more precise regarding the threshold c_2 :

- if $\mu = r$, then $c_2 = \frac{a\lambda^2\rho^2}{b(1-\rho^2)}$;
- if $\mu \neq r$ and $\rho\lambda < 0$, then no further condition on c_2 is needed;
- if $\mu \neq r$ and $\rho\lambda = 0$, then $c_2 = \frac{(\mu-r)^2b}{(a-\sigma)(1-\rho^2)}$;

It is rather interesting to compare this result with those of Proposition 3.4 and Proposition 3.5. Indeed, when $b > 0$, if $f(t) \equiv t$ then the stock price process does not satisfy an average squared market price of risk γ_1 above the threshold $a\lambda^2/b$. However, when $t/f(t)$ tends to infinity, then S has an average squared market price of risk γ_1 above the threshold $a\lambda^2/b$. When $b > 0$, $\lambda\rho \neq 0$ and $\mu = r$, if $f(t) \equiv t$ then the stock price process does not satisfy an average squared market price of risk γ_2 above the threshold $\frac{a\lambda^4\rho^2}{b(1-\rho^2)}$, but does so above the threshold $\frac{a\lambda^2\rho^2}{b(1-\rho^2)}$ when $t/f(t)$ tends to infinity. Finally, when $b > 0$, $\lambda\rho = 0$ and $\mu \neq r$ the stock price process never satisfies an average squared market price of risk γ_2 with speed $f(t) \equiv t$, but does above the threshold $\frac{b(\mu-r)^2}{(1-\rho^2)(a-\sigma)}$ whenever $t/f(t)$ tends to infinity.

Proof of Proposition 3.9. Let f be as stated in the proposition. For $b > 0$, the variance process is ergodic and its stationary distribution is a Gamma law with shape parameter a/σ and scale parameter σ/b (see [15]). In particular, $t^{-1} \int_0^t V_s ds$ converges in probability to a/b as t tends to infinity, and hence for

any $c_1 \in (0, a\lambda^2/b)$,

$$(3.6) \quad \lim_{t \nearrow +\infty} \mathbb{P} \left(\frac{1}{t} \int_0^t \gamma_1^2(s) ds < c_1 \right) = 0, \quad \text{and hence} \quad \lim_{t \nearrow +\infty} \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \gamma_1^2(s) ds < c_1 \right) = 0,$$

which proves the first part of the proposition.

Consider now γ_2 . When $\mu = r$, the definitions (2.3) implies that $\gamma_2 = -\rho\gamma_1/\sqrt{1-\rho^2}$, and hence

$$\lim_{t \nearrow +\infty} \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \gamma_2^2(s) ds < c_2 \right) = \lim_{t \nearrow +\infty} \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \gamma_1^2(s) ds < \frac{(1-\rho^2)c_2}{\rho^2} \right)$$

is equal to zero if and only if $(1-\rho^2)c_2/\rho^2 \in (0, a\lambda^2/b)$, and the proposition follows.

We now assume that $\mu \neq r$. If $a > \sigma$ we further know that (see proposition 4 in [1]) $t^{-1} \int_0^t V_s^{-1} ds$ converges in probability to $b/(a-\sigma)$ as t tends to infinity. Therefore for any $c \in (0, b/(a-\sigma))$ we have

$$(3.7) \quad \lim_{t \nearrow +\infty} \mathbb{P} \left(\frac{1}{t} \int_0^t \frac{ds}{V_s} < c \right) = 0, \quad \text{and hence} \quad \lim_{t \nearrow +\infty} \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \frac{ds}{V_s} < c \right) = 0.$$

Let c_2, c'_1, c'_2 be three strictly positive numbers such that $c_2 = c'_1 + c'_2$. The definition of γ_2 in (2.3) implies

$$\begin{aligned} \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \gamma_2^2(s) ds < c_2 \right) &= \mathbb{P} \left(\frac{1}{f(t)} \frac{(\mu-r)^2}{1-\rho^2} \int_0^t \frac{ds}{V_s} - \frac{2\rho\lambda(\mu-r)}{1-\rho^2} \frac{t}{f(t)} + \frac{1}{f(t)} \frac{\lambda^2\rho^2}{1-\rho^2} \int_0^t V_s ds < c_2 \right) \\ &\leq \mathbb{P} \left(\frac{1}{f(t)} \frac{\lambda^2\rho^2}{1-\rho^2} \int_0^t V_s ds < c'_1 \right) + \mathbb{P} \left(\frac{1}{f(t)} \frac{(\mu-r)^2}{1-\rho^2} \int_0^t \frac{ds}{V_s} - \frac{2\rho\lambda(\mu-r)}{1-\rho^2} \frac{t}{f(t)} < c'_2 \right) \\ &= \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \gamma_1^2(s) ds < c_1 \right) + \mathbb{P} \left(\frac{1}{f(t)} \int_0^t \frac{ds}{V_s} < \frac{1-\rho^2}{(\mu-r)^2} \left[c'_2 + \frac{2\rho\lambda(\mu-r)}{1-\rho^2} \frac{t}{f(t)} \right] \right) \end{aligned}$$

with $c'_1 = \frac{\rho^2}{1-\rho^2} c_1 > 0$. As long as $c_1 \in (0, a\lambda^2/b)$, the first probability tends to zero as t tends to infinity by (3.6). Now, when $\rho\lambda(\mu-r) < 0$, then since $t/f(t)$ tends to infinity, the second probability tends to zero (as t tends to infinity) by (3.7) because $c'_2 + \frac{2\rho\lambda(\mu-r)}{1-\rho^2} \frac{t}{f(t)}$ tends to $-\infty$ (and because the variance process is non-negative almost surely). No condition on c'_2 is needed here.

When $\rho\lambda = 0$, then the first line of the equation above simplifies to

$$\mathbb{P} \left(\frac{1}{f(t)} \int_0^t \gamma_2^2(s) ds < c_2 \right) = \mathbb{P} \left(\frac{1}{f(t)} \frac{(\mu-r)^2}{1-\rho^2} \int_0^t \frac{ds}{V_s} < c_2 \right).$$

From (3.7), it tends to zero as t tends to infinity when $0 < c_2 < \frac{(\mu-r)^2 b}{(1-\rho^2)(a-\sigma)}$, and hence the proposition follows from Definition 2.3. \square

We now state and prove our final result, namely a strong asymptotic arbitrage statement for the stock price process when the speed is sublinear. However it is not as clear as in Theorem 3.7 how to prove the exponentially decaying failure probability.

Proposition 3.11. *Assume that $a > \sigma$ and $\lambda\rho(\mu-r) \leq 0$, and let $\varepsilon \in (0, 1)$, $\gamma > 0$. Then S satisfies a strong asymptotic arbitrage with speed $f(t)$ and with $(\varepsilon_1, \varepsilon_2) = (\varepsilon, e^{-\gamma f(t)})$,*

- if and only if $\lambda \in \mathbb{R} \setminus \left[-\sqrt{\frac{2b\gamma(1-\rho^2)}{a\rho^2}}, \sqrt{\frac{2b\gamma(1-\rho^2)}{a\rho^2}} \right]$ when $\mu = r$ and $\rho^2 \leq 1/2$;
- if and only if $\lambda \in \mathbb{R} \setminus \left[-\sqrt{2b\gamma/a}, \sqrt{2b\gamma/a} \right]$ when $\mu = r$ and $\rho^2 \geq 1/2$;
- if $\mu \neq r$ and $\rho\lambda < 0$;
- if $\mu \neq r$, $\rho\lambda = 0$ and $\gamma < \frac{(\mu-r)^2 b}{2(a-\sigma)(1-\rho^2)}$.

Proof. Recall that we are in the framework of Proposition 3.9, so that $c_1 > 0$ and $c_2 > 0$ are the thresholds for γ_1 and γ_2 above which S has an average squared market price of risk. In this proof, we follow steps similar to those in [8]. For any $\varepsilon > 0$, fix $0 < \gamma < \bar{\gamma} < \gamma' < c_1/2 = \frac{a\lambda^2}{2b}$ and $t_0 > 8\gamma'/[(\gamma' - \gamma + \bar{\gamma})^2\varepsilon]$ such that for any $t \geq t_0$ we have $\mathbb{P}\left(f(t)^{-1} \int_0^t \gamma_1^2(s)ds \leq 2\gamma'\right) < \varepsilon/4$. Define the stopping time $\tau_1 := t \wedge \inf\{s \in [0, t] : \int_0^s \gamma_1^2(u)du \geq 2\gamma'f(t)\}$. Using the fact that $\int_0^{\tau_1} \gamma_1^2(s)ds \leq 2\gamma'f(t)$, Chebychev's inequality implies

$$\mathbb{P}\left(\left|\int_0^{\tau_1} \gamma_1(s)dW_1(s)\right| \geq (\gamma' - \gamma + \bar{\gamma})f(t)\right) \leq \frac{2\gamma'}{(\gamma' - \gamma + \bar{\gamma})^2 f(t)} < \frac{\varepsilon}{4}.$$

For $Z_{\tau_1} := \exp\left(-\int_0^{\tau_1} \gamma_1(s)dW_1(s) - \frac{1}{2}\int_0^{\tau_1} \gamma_1^2(s)ds\right)$, we then obtain

$$\begin{aligned} \mathbb{P}\left(Z_{\tau_1} \geq e^{(\bar{\gamma}-\gamma)f(t)}\right) &= \mathbb{P}\left(-\int_0^{\tau_1} \gamma_1(s)dW_1(s) - \frac{1}{2}\int_0^{\tau_1} \gamma_1^2(s)ds \geq (\bar{\gamma} - \gamma)f(t)\right) \\ &\leq \mathbb{P}\left(\left|\int_0^{\tau_1} \gamma_1(s)dW_1(s)\right| \geq (\bar{\gamma} - \gamma + \gamma')f(t)\right) + \mathbb{P}\left(\frac{1}{2}\int_0^{\tau_1} \gamma_1^2(s)ds \leq \gamma'f(t)\right) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

Take now $0 < \bar{\gamma} < \gamma'' < c_2/2$, and $t_1 > \frac{8\gamma''}{(\gamma'' - \bar{\gamma})^2\varepsilon}$ such that, for $t \geq t_1$, $\mathbb{P}\left(f(t)^{-1} \int_0^t \gamma_2^2(s)ds \leq 2\gamma''\right) < \varepsilon/4$. Define the stopping time τ_2 by $\tau_2 := t \wedge \inf\{s \in [0, t] : \int_0^s \gamma_2^2(u)du \geq 2\gamma''f(t)\}$ and the random variable $Z_{\tau_2} := \exp\left(-\int_0^{\tau_2} \gamma_2(s)dW_2(s) - \frac{1}{2}\int_0^{\tau_2} \gamma_2^2(s)ds\right)$. We then have $\mathbb{P}(Z_{\tau_2} \geq e^{-\bar{\gamma}f(t)}) \leq \varepsilon/2$. The two sets $A_t := \{Z_{\tau_1} \geq e^{(\bar{\gamma}-\gamma)f(t)}\} \in \mathcal{F}_t$ and $B_t := \{Z_{\tau_2} \geq e^{-\bar{\gamma}f(t)}\} \in \mathcal{F}_t$ clearly satisfy the following inequalities:

$$\begin{aligned} \mathbb{P}(A_t) &\leq \varepsilon/2, & \mathbb{Q}(A_t^c) &\leq e^{(\bar{\gamma}-\gamma)f(t)}, \\ \mathbb{P}(B_t) &\leq \varepsilon/2, & \mathbb{Q}(B_t^c) &\leq e^{-\bar{\gamma}f(t)}, \end{aligned}$$

where again we define the probability $\mathbb{Q}(B) := \mathbb{E}(BZ_t)$, for any $B \in \Omega$. Combining these inequalities, we conclude that there exist $t_0, t_1 > 0$ such that for $t \geq t_0 \vee t_1$, we have $\mathbb{P}(A_t \cup B_t) \leq \varepsilon$ and $\mathbb{Q}(A_t^c \cap B_t^c) \leq e^{-\gamma f(t)}$. Using [8, Proposition 2.1], we can now introduce the random variable $Y_t = -e^{-\gamma f(t)}\mathbf{1}_{A_t \cup B_t} + (1 - e^{-\gamma f(t)})\mathbf{1}_{A_t^c \cap B_t^c}$. Clearly $Y_t \in K_t$ and satisfies $Y_t \geq -e^{-\gamma f(t)}$ and $\mathbb{P}(Y_t \geq 1 - e^{-\gamma f(t)}) \geq 1 - \varepsilon$. Letting $\bar{t} := t_0 \vee t_1$, the proposition follows.

Note that the constraint $a\lambda^2/b = c_1 > 2\gamma$ reads $\lambda \in \mathbb{R} \setminus \left[-\sqrt{2b\gamma/a}, \sqrt{2b\gamma/a}\right]$. The constraints on c_2 depend on the sign of $\lambda\rho(\mu - r)$, as explained in Remark 3.10:

- if $\mu = r$ and $\rho^2 < 1/2$, then $c_1 > c_2$; then $c_2/2 > \gamma$ if and only if $\lambda \in \mathbb{R} \setminus \left[-\sqrt{\frac{2b\gamma(1-\rho^2)}{a\rho^2}}, \sqrt{\frac{2b\gamma(1-\rho^2)}{a\rho^2}}\right]$;
- if $\mu = r$ and $\rho^2 > 1/2$, then $c_1 < c_2$; then $c_1/2 > \gamma$ if and only if $\lambda \in \mathbb{R} \setminus \left[-\sqrt{2b\gamma/a}, \sqrt{2b\gamma/a}\right]$;
- if $\mu \neq r$ and $\rho\lambda < 0$, no further assumption on λ is needed;
- if $\mu \neq r$ and $\rho\lambda = 0$, then the following constraint has to hold: $0 < \gamma < \frac{(\mu-r)^2 b}{2(a-\sigma)(1-\rho^2)}$.

□

APPENDIX A. LARGE DEVIATIONS RESULTS

Proof of Lemma 3.2. Recall from [16, Chapter 6, Proposition 2.5] that for any $t \geq 0$, $\log \mathbb{E} \left(e^{X_t^{\alpha, \beta, 0}} \right) = -a\phi_{-\alpha, -\beta}(t) - \psi_{-\alpha, -\beta}(t)V_0$, where

$$\begin{aligned}\phi_{\alpha, \beta}(t) &:= -\frac{1}{\sigma} \log \left(\frac{2\chi e^{t(b-\chi)/2}}{2\sigma\alpha(1-e^{-\chi t}) + (\chi-b)e^{-\chi t} + (\chi+b)} \right), \\ \psi_{\alpha, \beta}(t) &:= \frac{\alpha[(\chi+b)e^{-\chi t} + (\chi-b)] + 2\beta(1-e^{-\chi t})}{2\sigma\alpha(1-e^{-\chi t}) + (\chi-b)e^{-\chi t} + (\chi+b)},\end{aligned}$$

with $\chi := \sqrt{b^2 + 4\sigma\beta}$. For any $t > 0$, the moment generating function of $X_t^{\alpha, \beta, 0}/t$ therefore reads $\Lambda_t^{\alpha, \beta}(u) := \mathbb{E} \left(e^{uX_t^{\alpha, \beta, 0}/t} \right)$, for $u \in \mathcal{D}_t^\beta$ where $\mathcal{D}_t^\beta = \left(-\infty, \frac{b^2 t}{4\sigma\beta} \right]$ if $\beta > 0$ and $\left[\frac{b^2 t}{4\sigma\beta}, +\infty \right)$ if $\beta < 0$. Straightforward computations yield

$$\Lambda^\beta(u) := \lim_{t \nearrow +\infty} t^{-1} \log \Lambda_t^{\alpha, \beta}(ut) = \frac{a}{2\sigma} \left(b - \sqrt{b^2 - 4\sigma\beta u} \right),$$

as given in (3.3), for $u \in \mathcal{D}_\beta := \lim_{t \nearrow +\infty} \mathcal{D}_t^\beta$, defined in (3.2). We further have

$$\partial_u \Lambda^\beta(u) = \frac{\beta a}{\sqrt{b^2 - 4\sigma\beta u}} \quad \text{and} \quad \partial_{uu} \Lambda^\beta(u) = \frac{2\beta^2 \sigma a}{(b^2 - 4\sigma\beta u)^{3/2}},$$

for any $u \in \mathcal{D}_\beta^o$, and hence $\partial_u \Lambda^\beta(\mathcal{D}_\beta^o) = \mathbb{R}_+^*$ if $\beta > 0$ and \mathbb{R}_-^* if $\beta < 0$. Therefore Λ^β is convex on \mathcal{D}_β , and hence the Gärtner-Ellis theorem (see [6]) applies, albeit only on subsets of $\partial_u \Lambda^\beta(\mathcal{D}_\beta^o)$. We now characterise the rate function Λ^* . Recall that the Fenchel-Legendre transform of Λ is defined by $\Lambda_\beta^*(x) := \sup \{ ux - \Lambda^\beta(u) : u \in \mathcal{D}_\beta \}$. Let us first consider the case $\beta > 0$. Since the function Λ' is strictly increasing on \mathcal{D}_β^o and $\partial_u \Lambda^\beta(\mathcal{D}_\beta^o) = \mathbb{R}_+^*$, then for any $x > 0$, the equation $\partial_u \Lambda^\beta(u) = x$ has a unique solution $u^*(x) = (b^2 x^2 - \beta^2 a^2) / (4\sigma\beta x^2)$, and we deduce $\Lambda_\beta^*(x) = u^*(x)x - \Lambda^\beta(u^*(x)) = (bx - a\beta)^2 / (4\sigma\beta x)$, for any $x > 0$. For $x \leq 0$, the definition of the Fenchel-Legendre transform implies $\Lambda_\beta^*(x) = +\infty$. In the case $\beta < 0$, an analogous analysis holds: the Gärtner-Ellis theorem applies on subsets of $\Lambda'(\mathcal{D}_\beta^o) = \mathbb{R}_-^*$ with rate function Λ_β^* given in (3.5) on \mathbb{R}_-^* and infinity on \mathbb{R}_+ . \square

Proof of Lemma 3.3. We first start with the case $b \neq 0$. The moment generating function of the random variable $X_t^{\alpha, \beta, \delta}/t$ is given by (see [2, proposition 2]),

$$\begin{aligned}\Lambda_t(u) &= \mathbb{E} \left(\exp \left(\frac{\alpha u}{t} V_t + \frac{\beta u}{t} \int_0^t V_s ds + \frac{\delta u}{t} \int_0^t V_s^{-1} ds \right) \right) \\ &= \frac{\Gamma(\kappa + \nu/2 + 1/2)}{\Gamma(\nu + 1)} \exp \left\{ \frac{b}{2\sigma} (at + V_0) - \frac{AV_0}{2\sigma} \coth \left(\frac{At}{2} \right) \right\} \\ &\quad \times \left(\frac{AV_0}{2\sigma \sinh(At/2)} \right)^{\nu/2 + 1/2 - \kappa} \left(\left(b - \frac{2\sigma\alpha u}{t} \right) \frac{\sinh(At/2)}{A} + \cosh(At/2) \right)^{-\nu/2 - 1/2 - \kappa} \\ &\quad \times {}_1F_1 \left(\kappa + \frac{\nu + 1}{2}, \nu + 1, \frac{A^2 V_0}{2\sigma \sinh(At/2) \left((b - \frac{2\sigma\alpha u}{t}) \sinh(At/2) + \cosh(At/2) \right)} \right)\end{aligned}$$

where $\kappa := \frac{a}{2\sigma}$, $A := \sqrt{b^2 - \frac{4\sigma\beta u}{t}}$, $\nu := \frac{1}{\sigma} \sqrt{(a - \sigma)^2 - \frac{4\sigma\delta u}{t}}$. The confluent hypergeometric function is defined by ${}_1F_1(u, v, z) = \sum_{n \geq 0} \frac{u^{(n)}}{v^{(n)}} \frac{z^n}{n!}$, with $v^{(n)}$ denoting the rising factorial $v^{(n)} := v(v+1) \dots (v+n-1)$.

As t tends to infinity, $t^{-1} \log \left(\frac{\Gamma(\kappa+\nu/2+1/2)}{\Gamma(\nu+1)} \right)$ clearly tends to zero and

$$\lim_{t \nearrow +\infty} \frac{1}{t} \log \left({}_1F_1 \left(\kappa + \frac{\nu+1}{2}, \nu+1, \frac{A^2 V_0}{2\sigma \sinh(At/2) \left[\left(b - \frac{2\sigma\alpha u}{t} \right) \sinh(At/2) + \cosh(At/2) \right]} \right) \right) = 0.$$

Therefore,

$$\begin{aligned} \Lambda^{\beta,\delta}(u) &:= \lim_{t \nearrow +\infty} t^{-1} \log \Lambda_t(tu) \\ &= \lim_{t \nearrow +\infty} \frac{1}{t} \left\{ \frac{b}{2\sigma} (at + V_0) - \frac{AV_0}{2\sigma} \frac{e^{At/2} + e^{-At/2}}{e^{At/2} - e^{-At/2}} + \left(\frac{\nu+1}{2} - \kappa \right) \log \left(\frac{AV_0}{\sigma (e^{At/2} - e^{-At/2})} \right) \right. \\ &\quad \left. - \left(\kappa + \frac{\nu+1}{2} \right) \log \left(\frac{b - 2\sigma\alpha u}{A} \left(\frac{e^{At/2} - e^{-At/2}}{e^{At/2} + e^{-At/2}} \right) + \frac{e^{At/2} + e^{-At/2}}{2} \right) \right\} \\ &= -\frac{\nu A}{2} - \frac{A}{2} + \frac{ba}{2\sigma} = \frac{ab}{2\sigma} - \frac{1}{2\sigma} \sqrt{((a-\sigma)^2 - 4\sigma\delta u)(b^2 - 4\sigma\beta u)} - \frac{1}{2} \sqrt{b^2 - 4\sigma\beta u}, \end{aligned}$$

for $u \in \mathcal{D}_{\beta,\delta}$ where the interval $\mathcal{D}_{\beta,\delta}$ is given in (3.2). We can then immediately compute

$$\partial_u \Lambda^{\beta,\delta}(u) = \frac{\sigma\beta}{\sqrt{b^2 - 4\sigma\beta u}} - \frac{8\sigma\delta\beta u - \beta(a-\sigma)^2 - \delta b^2}{\sqrt{((a-\sigma)^2 - 4\sigma\delta u)(b^2 - 4\sigma\beta u)}}, \quad \text{for any } u \in \mathcal{D}_{\beta,\delta}^o,$$

and hence

$$\partial_u \Lambda^{\beta,\delta}(\mathcal{D}_{\beta,\delta}^o) = \begin{cases} \mathbb{R}, & \text{if } \beta\delta < 0, \\ (2\sqrt{\delta\beta}, +\infty), & \text{if } \beta > 0, \delta > 0, \\ (-\infty, -2\sqrt{\delta\beta}), & \text{if } \beta < 0, \delta < 0. \end{cases}$$

We also have, for any $u \in \mathcal{D}_{\beta,\delta}^o$,

$$\partial_{uu} \Lambda^{\beta,\delta}(u) = \frac{2\sigma^2\beta^2}{(b^2 - 4\sigma\beta u)^{3/2}} + \frac{2\sigma(\delta b^2 - \beta(a-\sigma)^2)^2}{(((a-\sigma)^2 - 4\sigma\delta u)(b^2 - 4\sigma\beta u))^{3/2}}.$$

Therefore $\Lambda^{\beta,\delta}$ is strictly convex on $\mathcal{D}_{\beta,\delta}$, and the Gärtner-Ellis theorem (see [6]) only applies on subsets of $\partial_u \Lambda^{\beta,\delta}(\mathcal{D}_{\beta,\delta}^o)$. For any $x \in \partial_u \Lambda^{\beta,\delta}(\mathcal{D}_{\beta,\delta}^o)$, the equation $\partial_u \Lambda^{\beta,\delta}(u) = x$ has a unique solution $u^*(x)$ and hence $\Lambda_{\beta,\delta}^*(x) := \sup_{u \in \mathcal{D}_{\beta,\delta}} \{ux - \Lambda^{\beta,\delta}(u)\} = u^*(x)x - \Lambda^{\beta,\delta}(u^*(x))$.

We now move on to the case $b = 0$. From [2, Corollary 1], the limiting mgf of $X_t^{\alpha,\beta,\delta}$ reads

$$\Lambda^{\beta,\delta}(u) := \lim_{t \nearrow +\infty} t^{-1} \log \mathbb{E} \left(e^{u X_t^{\alpha,\beta,\delta}} \right) = -\sqrt{-\sigma\beta u} - \frac{1}{\sigma} \sqrt{-\sigma\beta u} \sqrt{(a-\sigma)^2 - 4\sigma\delta u},$$

for any $u \in \mathcal{D}_{\beta,\delta}$ where this interval now reads

$$\mathcal{D}_{\beta,\delta} = \begin{cases} \left[0, \frac{(a-\sigma)^2}{4\sigma\delta} \right], & \text{if } \beta < 0 \text{ and } \delta > 0, \\ \left[\frac{(a-\sigma)^2}{4\sigma\delta}, 0 \right], & \text{if } \beta > 0 \text{ and } \delta < 0, \\ \mathbb{R}_-, & \text{if } \beta > 0 \text{ and } \delta > 0, \\ \mathbb{R}_+, & \text{if } \beta < 0 \text{ and } \delta < 0. \end{cases}$$

Then

$$\partial_u \Lambda^{\beta,\delta}(u) = \frac{\sigma\beta}{2\sqrt{-\sigma\beta u}} + \frac{\beta\sqrt{(a-\sigma)^2 - 4\sigma\delta u}}{2\sqrt{-\sigma\beta u}} + \frac{2\delta\sqrt{-\sigma\beta u}}{\sqrt{(a-\sigma)^2 - 4\sigma\delta u}}, \quad \text{for any } u \in \mathcal{D}_{\beta,\delta}^o,$$

and hence

$$(A.1) \quad \partial_u \Lambda^{\beta, \delta}(\mathcal{D}_{\beta, \delta}^o) = \begin{cases} \mathbb{R}, & \text{if } \beta\delta < 0, \\ (2\sqrt{\delta\beta}, +\infty), & \text{if } \beta > 0 \text{ and } \delta > 0, \\ (-\infty, -2\sqrt{\delta\beta}), & \text{if } \beta < 0 \text{ and } \delta < 0. \end{cases}$$

We also have

$$\partial_{uu} \Lambda^{\beta, \delta}(u) = \frac{\sigma^2 \beta^2}{4(-\sigma\beta u)^{3/2}} - \frac{\beta(a - \sigma)^2}{4u\sqrt{(a - \sigma)^2 - 4\sigma\delta u}\sqrt{-\sigma\beta u}} - \frac{\sigma\beta\delta(a - \sigma)^2}{((a - \sigma)^2 - 4\sigma\delta u)^{3/2}\sqrt{-\sigma\beta u}}.$$

Clearly then, $\Lambda^{\beta, \delta}$ is convex on $\mathcal{D}_{\beta, \delta}$, and the Gärtner-Ellis theorem (see [6]) only applies on subsets of $\partial_u \Lambda^{\beta, \delta}(\mathcal{D}_{\beta, \delta}^o)$. For any $x \in \partial_u \Lambda^{\beta, \delta}(\mathcal{D}_{\beta, \delta}^o)$, the equation $\partial_u \Lambda^{\beta, \delta}(u) = x$ has a unique solution $u^*(x)$ and hence $\Lambda_{\beta, \delta}^*(x) := \sup_{u \in \mathcal{D}_{\beta, \delta}} \{ux - \Lambda^{\beta, \delta}(u)\} = u^*(x)x - \Lambda_{\beta, \delta}(u^*(x))$, and the lemma follows. \square

Lemma A.1. *For any $x \in \partial_u \Lambda^{\beta, \delta}(\mathcal{D}_{\beta, \delta}^o)$, the equation $\partial_u \Lambda^{\beta, \delta}(u^*(x)) = x$ admits a unique solution $u^*(x) \in \mathcal{D}_{\beta, \delta}^o$. The function $\Lambda_{\beta, \delta}^*$ is strictly convex and satisfies $\Lambda_{\beta, \delta}^*(x) = u^*(x)x - \Lambda^{\beta, \delta}(u^*(x))$ on $\partial_u \Lambda^{\beta, \delta}(\mathcal{D}_{\beta, \delta}^o)$ and is (positive) infinite outside. In the case $\beta\delta \geq 0$, $\Lambda_{\beta, \delta}^*$ is strictly positive. When $\beta\delta < 0$, $\Lambda_{\beta, \delta}^*$ admits a unique minimum, which is equal to zero (and is attained at the origin) if and only if $a > \sigma$.*

Proof. When $\beta\delta < 0$, the image of $\mathcal{D}_{\beta, \delta}^o$ by $\partial_u \Lambda^{\beta, \delta}$ is the whole real line, and the representation of $\Lambda_{\beta, \delta}^*$ in the lemma clearly follows. Now, suppose there exists $\bar{x} \in \mathbb{R}$ such that $\Lambda_{\beta, \delta}^*(\bar{x}) = 0$. Then there exists some (possibly non-unique) $u^*(\bar{x}) \in \mathcal{D}_{\beta, \delta}$ such that $u^*(\bar{x})\bar{x} = \Lambda^{\beta, \delta}(u^*(\bar{x}))$, i.e. $\Lambda^{\beta, \delta}(u^*(\bar{x}))/u^*(\bar{x}) = \bar{x}$. But $u^*(\bar{x})$ also satisfies $\partial_u \Lambda^{\beta, \delta}(u^*(\bar{x})) = \bar{x}$. A straightforward analysis shows that the equality $\partial_u \Lambda^{\beta, \delta}(u) = \Lambda^{\beta, \delta}(u)/u$ is satisfied if and only if $u = 0$ and $a > \sigma$.

When $\beta > 0$ and $\delta > 0$, for any $x \leq 2\sqrt{\beta\delta}$, the map $u \mapsto ux - \Lambda^{\beta, \delta}(u)$ is strictly decreasing on $\mathcal{D}_{\beta, \delta}^o$, and the result follows. By definition, the function $\Lambda_{\beta, \delta}^*$ admits a (unique) minimum \bar{x} if and only if (i) there exists $u(\bar{x}) \in \mathcal{D}_{\beta, \delta}$ such that $u(\bar{x})\bar{x} = \Lambda^{\beta, \delta}(u(\bar{x}))$ and (ii) $\Lambda^{\beta, \delta}(u) > u\bar{x}$ for any $u \in \mathcal{D}_{\beta, \delta} \setminus \{u(\bar{x})\}$. A straightforward analysis shows that the function $u \mapsto \Lambda^{\beta, \delta}(u)/u$ on \mathbb{R}_+ is strictly increasing and maps \mathbb{R}_+ to $(2\sqrt{\beta\delta}, +\infty)$. On $\mathbb{R}_+ \cap \mathcal{D}_{\beta, \delta}$, it is strictly increasing and maps this interval to $(-\infty, -2\sqrt{\beta\delta})$. Therefore the inequality $\Lambda(u) > ux$ holds if and only if both (a) $\Lambda^{\beta, \delta}(u)/u > x$ for $u \in \mathbb{R}_+ \cap \mathcal{D}_{\beta, \delta}$ and (b) $\Lambda^{\beta, \delta}(u)/u < x$ for $u < 0$. Case (b) clearly only holds for $x < 2\sqrt{\beta\delta}$, which is not valid. The other cases are treated analogously. The case $\beta\delta = 0$ is straightforward. \square

REFERENCES

- [1] M. Ben Alaya and A. Kebaier. Parameter estimation for the square root diffusions: ergodic and non ergodic case. *Stochastic Models*, 28 (4): 609-634, 2012.
- [2] M. Ben Alaya and A. Kebaier. Asymptotic behavior of the maximum likelihood estimator for ergodic and non ergodic square-root diffusions. Preprint, hal.archives-ouvertes.fr-00640053, 2011.
- [3] M.L.D. Mbele Bidima and M. Rasonyi. On long-term arbitrage opportunities in Markovian models of financial markets. *Annals of Operations Research*, 200 (1):131-146, 2012.
- [4] M. Craddock and K. A. Lennox. The calculation of expectations for classes of diffusion processes by Lie symmetry methods. *The Annals of Applied Probability*, 19 (1): 127-157, 2009.
- [5] F. Delbaen and W. Schachermayer. The Mathematics of Arbitrage. Springer Finance, 2006.
- [6] A. Dembo and O. Zeitouni. Large deviations techniques and applications. Jones and Bartlet publishers, Boston, 1993.
- [7] K. Du. and A. Neufeld. A note on asymptotic exponential arbitrage with exponentially decaying failure probability. Forthcoming in *Journal of Applied Probability*, 2013.

- [8] H. Föllmer and W. Schachermayer. Asymptotic arbitrage and large deviations. *Mathematics and Financial Economics*, 1 (34): 213-249, 2007.
- [9] J.P. Fouque, G. Papanicolaou, R. Sircar and K. Solna. Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives. CUP, 2011.
- [10] J. Gatheral. The Volatility Surface: a practitioner's guide. Wiley, 2006.
- [11] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*,(6): 327-342, 1993.
- [12] C.C. Heyde and B. Wong. On changes of measure in stochastic volatility models. *Journal of Applied Mathematics and Stochastic Analysis*, 2006, Article ID 18130, 2006.
- [13] Y. Kabanov and D. Kramkov. Asymptotic arbitrage in large financial markets. *Fin. & Stochastics*, 2: 143-172, 1998.
- [14] I. Klein and W. Schachermayer. Asymptotic arbitrage in non-complete large financial markets. *Theory of Probability and its Applications*, 41 (4): 927-934, 1996.
- [15] Y.A. Kutoyants. Statistical inference for ergodic diffusion processes. *Springer series in Statistics. Springer-Verlag London Ltd*, 2004.
- [16] D. Lamberton and B. Lapeyre. Introduction au calcul stochastique appliqué la finance, 2nd edition. Ellipses Édition Marketing, Paris, 1997.
- [17] R.C. Merton The theory of rational option pricing. *Bell.J.Econ.Manag.Sci*, 4 : 141-183, 1973.
- [18] D. Revuz and M. Yor. Continuous martingales and Brownian motion. Springer-Verlag, Berlin, 1999.
- [19] D.B. Rokhlin. Asymptotic arbitrage and numéraire portfolio in large financial markets. *Finance and Stochastics*, 12 (2): 173-194, 2008.

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